Direct Proof

A proof must work for *all* cases at once. Checking examples is not a proof; algebra is, because a letter stands for every integer simultaneously.

Fact (Representing integers) — For integer n:

an even number	$2n$
an odd number	$2n+1$ (or $2n-1$)
consecutive integers	$n,\ n+1,\ n+2$
consecutive even numbers	$2n,\ 2n+2$
consecutive odd numbers	$2n-1,\ 2n+1$
a multiple of k	kn

To prove a result is even, aim for $2\times$(integer); a multiple of 8, aim for $8\times$(integer).

Example

Prove that $(2n+1)^2-(2n+1)$ is even for all positive integers n.

Example

Prove that the difference between the squares of two consecutive odd numbers is divisible by 8.

Example

Prove that the square of any odd number is one more than a multiple of 8.

Example

1. Show that $(n-1)n(n+1) = n^3 - n$.
2. Hence prove that $n^3 - n$ is divisible by 6 for every integer n.

Example (Multiplying three brackets)

Expand and simplify $(x+1)(x+2)(x+3)$.

Textbook Exercises: SPS Course 0.1, Exercise 2

Disproof and Implication

Definition. A statement of the form "for all $n, \ldots$" is disproved by a single counterexample.

Example

Disprove: "$n^2 + n + 41$ is prime for every positive integer n."

Definition.

$$P \Rightarrow Q \text{ (}P\text{ implies }Q\text{)} \qquad P \Leftarrow Q \qquad P \iff Q \text{ (}P\text{ if and only if }Q\text{)}$$

Proving $P \Rightarrow Q$ says nothing about the converse $Q \Rightarrow P$.

Example

Insert $\Rightarrow$, $\Leftarrow$ or $\iff$ between each pair:

1. $x = 3$ $\qquad x^2 = 9$
2. $5x - 3 = 27$ $\qquad x = 6$
3. n is divisible by 4 $\qquad n^2$ is divisible by 4

Proof by Factorisation

Example

Prove that $n^2 - 1$ is never prime for any integer $n > 2$.

Example

Find all pairs of positive integers x, y with $x^2 - y^2 = 35$, proving you have found them all.

Example

Prove that two non-zero squares can never differ by 2.

Exercise. Find the last digit of $2^{333} + 7^{333}$. (Track the cycle of last digits of the powers of 2 and of 7.)

Exercise. Prove that the sum of any four consecutive integers is never divisible by 4.

Textbook Exercises: SPS Course 0.1, Exercises 1, 5B and Revision Exercise 0.1